DAB-003-2014008 Seat No. ______

B. Sc. (Sem. IV) (CBCS) Examination

April - 2022

Mathematics : Paper - IV (A)

(Linear Algebra, Real Analysis & Differential Geometry)

Faculty Code : 003
Subject Code : 2014008

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) Attempt any five questions.
(2) Figures to the right indicate full marks of the question.

1 (a) Answer the following questions in short : **4**

(1) Every convergent sequence is Cauchy. (True / False)

(2) A sequence $\left\{\frac{4^{3n}}{3^{4n}}\right\}$ ________.

(converges / diverges / oscillates)

(3) Define : Convergent sequence

(4) Define : Increasing sequence.

(b) Prove that every convergent sequence is bounded. **2**

(c) State Cauchy's general principle and hence using it prove that the sequence S_n cannot converge, where **3**

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n}.$$

(d) If $\lim_{n\to\infty} a_n = l$, then prove that **5**

$$\lim_{n\to\infty}\left(\frac{a_1 + a_2 + \ldots\ldots + a_n}{n}\right) = l$$

2 (a) Answer the following questions in short : 4

(1) Every monotonically bounded sequence is convergent. (True / False)

(2) $\lim_{n\to\infty}\left(1\frac{2}{1}\frac{3}{2}\frac{4}{3}\cdots\cdots\frac{n}{n-1}\right)^{\frac{1}{n}} = $ ________

(3) Define : Bounded sequence

(4) State : Bolzano-Weierstrass Theorem.

(b) Evaluate $\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}+\ldots\ldots+\frac{1}{\sqrt{n^2+n}}}\right]$ 2

(c) If $\{a_n\}$ is any sequence, then prove that inf 3

$a_n \le \underline{\lim}\, a_n \le \overline{\lim}\, a_n \le \sup a_n$.

(d) For convergent sequences $\{a_n\}$ and $\{b_n\}$, prove that 5

$\lim_{n\to\infty}(a_n+b_n)=\left(\lim_{n\to\infty}a_n\right)+\left(\lim_{n\to\infty}b_n\right)$ and

$\lim_{n\to\infty}(a_n b_n)=\left(\lim_{n\to\infty}a_n\right)\left(\lim_{n\to\infty}b_n\right)$

3 (a) Answer the following questions in short : 4

(1) For a positive term series $\sum a_n$ and $\sum b_n$, if

$\lim_{n\to\infty}\frac{a_n}{b_n}=0$, then

(a) both $\sum a_n$ and $\sum b_n$ converges

(b) both $\sum a_n$ and $\sum b_n$ diverges

(c) $\sum a_n$ converges $\Rightarrow \sum b_n$ converges

(d) $\sum b_n$ converges $\Rightarrow \sum a_n$ converges

(2) $\sum \frac{\cos(n\pi)}{n}$ is an alternating series. (True / False)

(3) State : D'Alembert's test.

(4) Define : Convergence of a series.

(b) Show that the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots+\frac{1}{2^{n-1}}+\ldots\ldots$ is convergent. **2**

(c) Prove that the necessary condition for convergence of an infinite series $\sum u_n$ is that $\lim_{n\to\infty} u_n = 0$. Discuss its converse with justification. **3**

(d) Test the convergence of the series **5**

$$\frac{\alpha}{\beta}+\frac{1+\alpha}{1+\beta}+\frac{(1+\alpha)(2+\alpha)}{(1+\beta)(2+\beta)}+\frac{(1+\alpha)(2+\alpha)(3+\alpha)}{(1+\beta)(2+\beta)(3+\beta)}+\ldots\ldots\ldots$$

4 (a) Answer the following questions in short : **4**

(1) The series $\sum_{n=0}^{\infty} x^n$ is convergence if and only if __________.

(2) Define : positive term series.

(3) Define : alternating series

(4) State p-test

(b) Test the convergence or divergence of the series **2**

$$\sum_{n=0}^{\infty} \frac{\pi^{n3}}{3^{n\pi}}$$

(c) Test the convergence or divergence of the series **3**

$$\frac{1}{4\cdot 6}+\frac{\sqrt{3}}{6\cdot 8}+\frac{\sqrt{5}}{8\cdot 10}+\frac{\sqrt{7}}{10\cdot 12}+\ldots\ldots\ldots$$

(d) Find the radius and interval of convergence for the series $\sum_{n=0}^{\infty} \frac{(-3x)^n}{\sqrt{n+1}}$. **5**

5 (a) Answer the following questions in short : 4

(1) Define : Linear Transformation.

(2) Any linear transformation from $\mathbb{R}^2$ to itself is onto. (True / False)

(3) If rank of linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is 2 then dimension of $\ker(T)$ is ______.

(4) A linear transformation T is one-one if and only if nullity of T is ______.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined as $T(x, y) = (x, x + y, y)$. Then find rank of T. 2

(c) Let $T : V \to V$ be any linear transformation such that $T^2 - T + 1 = 0$, then prove that T is non singular. 3

(d) Let $\{v_1, v_2, ..., v_n\}$ be a basis of vector space V and let $w_i; 1 \le i \le n$ be any set of (not necessarily distinct) vectors in vector space W. Then show that there is an unique linear transformation $T : V \to W$ such that $T(v_i) = w_i$. 5

6 (a) Answer the following questions in short : 4

(1) A map $T : P_n(\mathbb{R}) \to P_{n+1}(\mathbb{R})$ defined as $T(p(x)) = xp(x), \quad \forall p(x) \in P_n(\mathbb{R})$ is not linear transformation. (True / False)

(2) Define : Kernel of linear transformation.

(3) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ is an identity linear transformation then rank of T is ______.

(4) Define : idempotent linear transformation.

(b) Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be defined as $T(x, y, z) = (x - y + z, x + y - z)$. Then find nullity of T. **2**

(c) Find $T(2, 5, 7)$. Given that a linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^2$ such that $T(e_1) = (1, 1)$, $T(e_1 + e_2) = (1, 0)$, $T(e_1 + e_2 + e_3) = (1, -1)$, where $\{e_1, e_2, e_3\}$ standard basis of $\mathbb{R}^3$. **3**

(d) State and prove rank-nullity theorem. **5**

7 (a) Answer the following questions in short : **4**

(1) Define : Linear functional

(2) Define : Eigen vector of linear transformation

(3) Define : Eigen basis

(4) Every invertible matrix is diagonalizable. (True / False)

(b) Define : Adjoint of a linear transformation. **2**

(c) Let $T:P_3(\mathbb{R}) \to P_2(R)$ defined as $T(p(x)) = p'(x)$. Find matrix $[T:B_1, B_2]$ with respect to standard basis $B_1 = \{1, x, x^2, x^3\}$ and $B_2 = \{1, x, x^2\}$. **3**

(d) Let $T:V \to V$ be a linear transformation and let B be a basis of V. Then show that T is singular if and only if $\det[T, B] = 0$. **5**

8 (a) Answer the following questions in short : **4**

(1) Define : dual of vector space.

(2) Define : eigen value of linear transformation.

(3) Write sufficient condition in terms of eigen value for a linear transformation $T:V \to V$ with $\dim(V) = n$ to be diagonalizable.

(4) Every diagonalizable matrix is invertible. (True / False)

(b) Find eigen values of $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x, 2y, 3z)$ **2**

(c) Let $T = V \to V$ be a linear transformation and let B a basis of V. Then show that λ is an eigen value of T if and only if $\det\left[(T-\lambda I), B\right] = 0$, where $I = V \to V$ is the identity linear transformation and $\dim(V) = n$. **3**

(d) Find eigen values and eigen vectors for the linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x+y+3z, x+5y+z, 3x+y+z)$ by considering the standard basis of $\mathbb{R}^3$. **5**

9 (a) Answer the following questions in short : **4**

(1) Define : radius of curvature

(2) Define : point of inflexion

(3) What is isolated point to the curve ?

(4) Define : oblique asymptote.

(b) Find nature of the double point to the curve $x^3y + y^4 + x^3 - x^2 + \left(y + a^2\right)y^2 = 0; \quad a > 0$. **2**

(c) Show that the curve $x^2 - 4y = 0$ has no asymptotes. **3**

(d) Show that radius of curvature of cycloid $x = 2(t + \sin t)$, $y = 2(1 - \cos t)$ at any point is $8 \cot \frac{t}{2}$. **5**

10 (a) Answer the following questions in short : **4**

(1) If curve is concave downwards at a point then radius of curvature to the curve at the point is negative. (True / False)

(2) What is the curvature of curve $x^2 + y^2 = 4$?

(3) Define : asymptote.

(4) Define : cusp.

(b) Find Asymptotes parallel to the co-ordinate axis to the curve $x^2y^2 = 4\left(x^2 + y^2\right)$. **2**

(c) If curve passes through origin and Y axis is the tangent to the curve at origin, then show that radius of curvature to the curve at origin is $\lim_{(x,\, y)\to(0,\, 0)}\left(\frac{y^2}{2|x|}\right)$. **3**

(d) Obtain formula for radius of curvature of Cartesian curve $y = f(x)$. **5**
